

Five-parameter logistic functions for partition curves

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Abstract. A five-parameter logistic function (5PL) and a five-parameter generalised logistic function (5PGL) for partition generating symmetric and asymmetric partition curves are presented. For symmetric curves Whiten and Modified Whiten functions are routinely used in Coal Preparation to fit curves to partition data from laboratory and density tracer test results. The functions are also used in process simulations. Whiten functions produce curves that are rotationally symmetric about an inflection point. They cannot produce asymmetric curves that are skewed to product or rejects. The International Standard for density tracer testing (ISO 5146) introduced the 5PL function to cater for asymmetry that had been found in sets of partition data. However, it required modification for use in process simulations to prevent unintended changes in Epm when the cutpoint is changed. A 5PGL function has since been identified that can also manage asymmetry and be used in process simulations without modification. For both the 5PL and 5PGL functions the relative values of parameters need to be considered depending on if the partition is to rejects or to product, and if skewness is to rejects or to product. Process performance indicators derived from the functions are also presented.

1 Introduction

Partition curves for density separations are plots of relative density (RD) versus the percentage (the partition number, PNo) of the RD fractions in the feed to a density separator that report to product or rejects. The Tromp curve had its inception in 1937 when K. F. Tromp published a paper (G.J. Sanders [1]). At the time the curves were drawn by hand and plotting partition data on log probability or arithmetic probability paper was a common technique used in Europe (Peng, F. F. et al. [2]) to enable comparison of curves generated from testing different separators. Peng also lists different methods of transforming Tromp curves for use in computer-based process simulations. One of the methods quoted was - “using the probable error or the imperfection to describe the linear portion of the curve and an interpolation technique for the nonlinear portion.” The references given for this method date from 1966 to 1976. G. J. Sanders [1] in 1978 produced a partition model for use in a process simulation computer program that included a straight line function for the central section of the curve and hyperbolic functions for the curved sections either side of the central section. AS 5213

[3] notes that W.J. Whiten developed a function that appeared in a publication in 1996 [4], that could be used for the whole curve.

The Whiten function has proven particularly useful for fitting curves to data from sampling and testing programs as well as density tracer testing. It is widely used in modelling density separation and in process simulations. Equation 1 is the Whiten function for partition to rejects.

$$PNo = \frac{100}{\left(1 + e^{\frac{\ln(3)}{Epm}(RD_{50}-RD)}\right)} \quad (1)$$

Where:

RD = the relative density variable.

RD₅₀ = Relative density corresponding to a PNo of 50, the RD of the cutpoint and the inflection point of the curve.

Epm = the mean probable error, the average of RD₅₀-RD₂₅ and RD₇₅-RD₅₀. (The probable error value could be calculated as RD₅₀-RD₂₅ or RD₇₅-RD₅₀ for symmetric curves. The average value method was adopted when in the early use of partition curves, asymmetry became evident (Peng [2]).)

Ln(3)/Epm determines how quickly the curve extends from the minimum to maximum partition number, i.e. from 0 to 1, or 0 to 100%. It is essentially the rate or slope parameter for the curve function.

The Whiten function is based on a generalised logistic function, equation 2.

$$f(x) = \frac{1}{(1 + e^{b(c-x)})} \quad (2)$$

Where:

b = rate parameter, which can be a positive or negative value.

c = the x value of the point of inflection, the midpoint of the curve.

Equation 1 is unsuitable for fitting data where a proportion of the feed coal is short-circuited to product and/or rejects streams (Sherritt [5]). A function for this data needs partition number asymptotes of less than one hundred and/or greater than zero to fit the data. The Whiten function was changed to include parameters for tails [Vince 6]. The modified function is a 4 parameter generalised logistic, 4PGL, of the form given in equation 3.

$$PNo = a + \frac{(d-a)}{\left(1 + e^{\frac{\ln(3)}{Epm}(RD_{50}-RD)}\right)} \quad (3)$$

Where a and d define the low and high limits, asymptotes, of the curve. If the low limit is not zero and/or the high limit is not 100, then the curve is said to have a “tail” or “tails”. The rate parameter, ln(3)/Epm, can be made positive or negative depending on if the curve required is for partition to product or to rejects.

Figure 1 shows four partition to rejects curves based on equation 3. The nominal Epm value used in each case to generate the rate parameter was 0.005. Cutpoints were adjusted to separate the curves, and different combinations of a and d parameters were used as shown in

the legend. Table 1 shows that the nominal Epm value only applies where a and d parameters are 0 and 100, respectively. The nominal Cutpoint (RD50) used in the function is the actual cutpoint only when a and d are 0 and 100, or a = 100-d. In all other instances the RD50 and Epm will have to be determined from the resultant curve. The inflection point's relative density (RD) remains the same as the nominal RD50 value used in the equation. The curves between the asymptote limits are all symmetrical about their inflection points. Hence the function is not suitable for fitting to skewed data.

Given the potential for confusion with using Epm and RD50 as parameters in partition functions, “b” will be used instead of $\ln(3)/Epm$ for the rate parameter in this paper, and

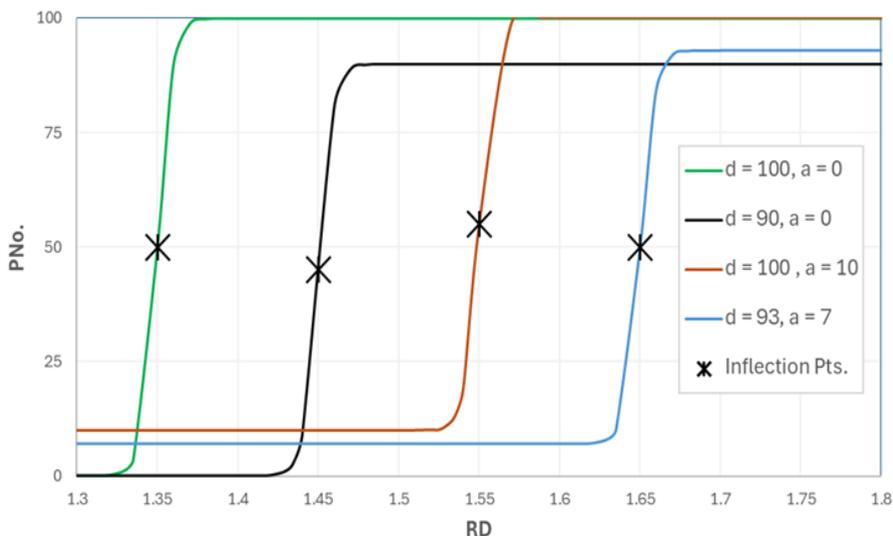


Fig. 1. Partition to rejects with different combinations of a and d parameters.

RD50 will be replaced with “c”, unless otherwise stated.

Table 1. Variation of curve properties.

(The Epm used to calculate the b parameter is 0.005.)

Curve	Parameter				Results			
	a	b	c	d	Epm	RD ₅₀	Inflection Pt. Coordinates	
							RD (c)	PNo.
1	100	109.86	1.35	0	0.0050	1.350	1.35	50
2	90	109.86	1.45	0	0.0058	1.451	1.45	45
3	100	109.86	1.55	00	0.0058	1.549	1.55	55
4	93	109.86	1.65	7	0.0067	1.650	1.65	50

2 Five-parameter partition functions

Density tracer testing has highlighted the necessity for partition functions to address asymmetry in test data. Asymmetry is present when the curve is skewed to the left or the right of the inflection point, as illustrated in Figure 2. If the skewness is to the left, it is “skewed to product”; if it is to the right, it is “skewed to rejects.” Two functions are presented that incorporate the standard four parameters for defining the partition curve, along with an

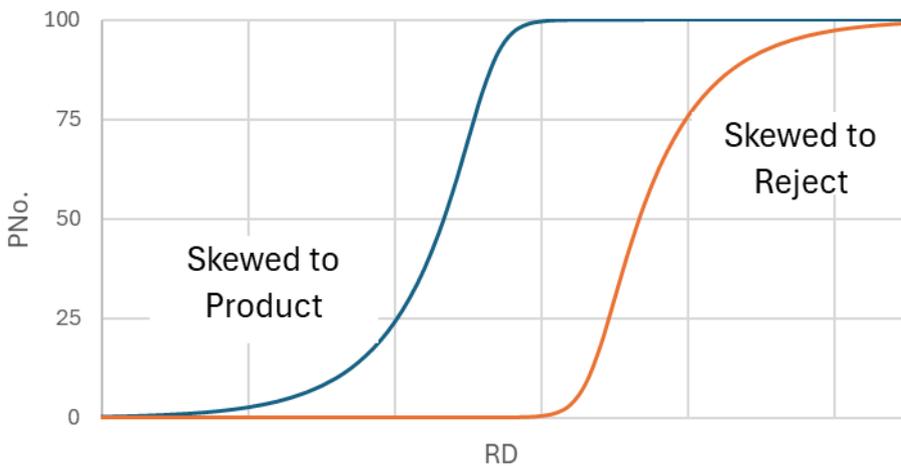


Fig. 2. Asymmetric partition curves - skewness types.

additional parameter for asymmetry. These functions are the five-parameter logistic function (5PL) and the five-parameter generalised logistic function (5PGL).

2.1 Five-parameter logistic function, 5PL

Figure 3 shows an asymmetric curve fitted to data points using 5PL, equation 4, that is given in ISO 5146 [7].; The symbols for parameters “a” and “d” have been exchanged, compared to the ISO Standard version, but their values remain unchanged, i.e., in this case, the a parameter is the upper limit of the curve and d is the lower limit. In some cases, the reverse can apply, i.e., a becomes the lower limit and d the upper limit. There are four possible permutations of a and d in the equation. The parameter naming arrangement used here aids the comparison of the five-parameter functions in this paper.

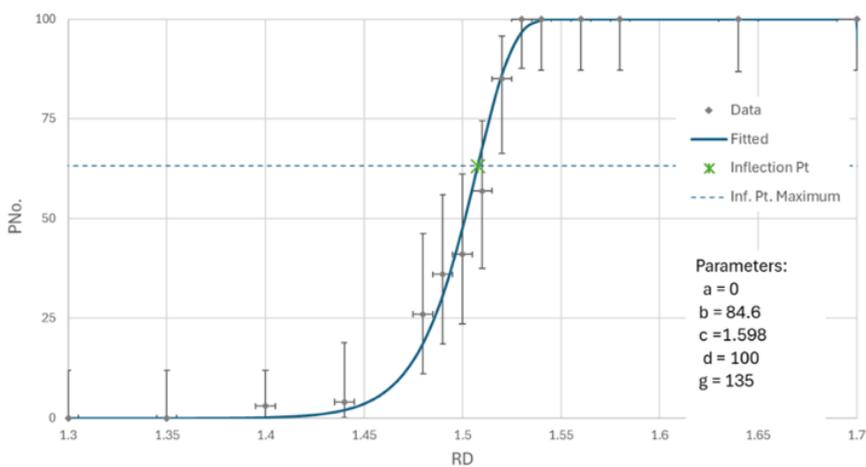


Fig. 3. Curve fitting by 5PL.

$$PNo = a + \frac{d-a}{\left(\left(1+\frac{x}{c}\right)^b\right)^g} \tag{4}$$

Where for partition to rejects:

a = the upper limit of the curve

d = the lower limit of the curve.

b = slope or rate parameter

c = the x value (RD) of the inflection point

g = asymmetry parameter

x = relative density variable

Equation 5 gives the relative density for a given partition number for the 5PL function.

$$RD_{PNo} = c \left(\left(\frac{a-d}{PNo-d} \right)^{\frac{1}{g}} - 1 \right)^{\left(\frac{1}{b} \right)} \quad (5)$$

2.2 Five-parameter generalised logistic function, 5PGL

An alternative to 5PL has been identified by the author since the publication of the ISO standard. The function, equation 6, is an adaption of the Richards curve [8].

$$PNo = a + \frac{(d-a)}{\left(1 + \frac{1}{g} e^{b(x-c)} \right)^g} \quad (6)$$

Where for partition to rejects, the parameters are the same as for the 5PL function given above. With $g = 1$ and b set to positive, equation 6 is effectively the same as the 4PGL Whiten function, equation 3.

The inverse of equation 6 for determining the RD of a partition number is given in equation 7.

$$RD = c - \frac{1}{b} \ln \left(g \left(\left(\frac{(d-a)}{(PNo-a)} \right)^{\frac{1}{g}} - 1 \right) \right) \quad (7)$$

Substituting the b parameter with $\ln(3)/Epm$ as used in the Whiten functions (equations 1 and 3) enables equation 6 to replace the Whiten functions in process simulations where an algorithm adjusts the Epm value in the model to produce the expected Epm value resulting from changing the cutpoint of the separation.

2.3 Characterisation of 5PL and 5PGL partition functions

2.3.1 Inherent asymmetry – 5PL

Setting $g = 1$ for 5PL makes it comparable to 4PGL (modified Whiten). 5PL is based on the ratio of particle relative density to c , the relative density of the inflection point and 4PGL is based on the difference between the two. In this situation, 4PGL produces a symmetric curve, whereas 5PL exhibits a minor amount of asymmetry. From a coal preparation perspective, the level of this inherent asymmetry is insignificant. The absolute value of the differences of relative densities for partition number pairs, (25, 50) and (75, 50) for $Epm = 0.001$ to 0.01 ranges from 0.067% to 0.67%, respectively. The level of asymmetry can be reduced further, if required, by marginally reducing the g value below 1.

2.3.2 Variation of cutpoint - 5PL

5PL was evaluated by changing cutpoints as would be occurring in process simulations and it was found that as the cutpoint is changed there may be a notable change in the Epm of the resultant curves (Figure 4). Given that partition functions are used in modelling of separations and in process simulations where cutpoint is a key variable, it would be necessary to be able to change parameters in 5PL so that Epm values at different cutpoints match what is expected in practice when cutpoints are changed. Process simulations based on Whiten functions

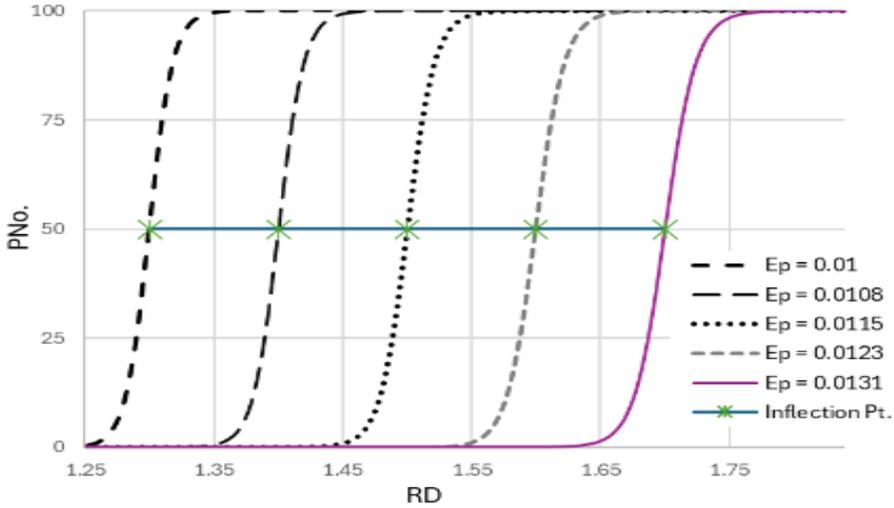


Fig. 4. For 5PL the E_p changes as the cutpoint is changed.

normally adjust E_{pm} via an algorithm. Initial work with 5PL during preparation of ISO 5146 produced a numerical solution to be applied to the b parameter to mitigate, minimise, the change so that existing simulation algorithms could still be used [9]. However, the factor does not eliminate the error entirely.

An alternative solution, to the issue of changing E_{pm} with cutpoint has since been developed by the author by solving for b in the inverse 5PL function, Equation 7, used in calculating E_{pm} . Equation 8 gives the formula that may be used to replace b in equation 6.

$$b = \frac{\ln(3)}{\ln\left(\frac{E_{ps} + \sqrt{E_{ps}^2 + c^2}}{c}\right)} \tag{8}$$

The E_{ps} value used in equation 8 is the E_{pm} of symmetric versions of the curve without tails, and the actual E_{pm} of the resultant curve has to be calculated when tails are present. There is no algebraic solution to include g in the function, it can only be done using numerical methods for each combination of g and c .

The formula includes the inflection parameter c and adjusts the value for b if a different cutpoint is required while preserving the E_{pm} . This function may be used with existing process simulation models without changing E_{pm} adjustment algorithms. It may also be used when the asymmetry parameter, g , does not equal 1. When $g = 1$ there is no notable change in resultant E_{pm} for different E_{ps} values and values from 1.3 to 2.0. Table 2 shows expected changes for $g = 0.02, 0.2,$ and 200 for the same density range. The results show the minimum and maximum change in E_{pm} and indicates that any changes in c are unlikely to be significant except for extremely poor E_{pm} 's and very low g values (i.e., with a very wide level of skewness).

Given the above findings the revised 5PL function has potential for use in process simulations as well as general curve fitting and process performance measurement.

Table 2. Difference in calculated Epm using $c = 1.30$ and 1.70 .

(Eps is the Ep value used in calculating the b parameter for the function.)

Eps	g	Resultant Epm		Change %
		c = 1.30	c = 1.70	
0.01	0.02	0.1877	0.2005	+6.9
0.001	0.02	0.0243	0.0244	+0.69
.01	0.2	0.0255	0.0257	+1
.001	0.2	0.0026	0.0026	+0.1
.01	200	0.0075	0.0074	-1.4
.001	200	0.0007	0.0007	-0.14

2.4 Comparison of 5PL and 5PGL

Figures 5a, 5b, 6 and 7 demonstrate the characteristics of the curves produced by the five-parameter functions when one parameter is varied and others are kept constant. The first two, figures 5a and 5b show the effects of changing the b parameter and the c parameter values, respectively. This is equivalent to changing the Epm and cutpoint. In this situation $a = 100$, $d = 0$ and $g = 1$. Both functions exhibit similar behaviour.

Figure 6 shows the effects of changing the a and d parameters that determine the asymptotes of the curve. Changing these parameters, changes the vertical domain of the curve and may be applicable when low and /or high density material short circuit to the wrong product stream and produces “tails,” i.e. over-dense material to the clean coal product stream or under-dense material to the rejects. This is likely to happen when the spigot or the vortex finder is overloaded. Because the vertical domain of the curve has changed, the inflection point also changes in proportion to the net size of the tails. Again, both functions exhibit similar behaviour.

As noted previously for the 4 parameter Whiten function, if an RD_{50} value is used in the functions, it will not be equal to the RD_{50} of the resultant curve unless the two tails of the same proportion. The RD of the inflection point does not change as a and/or d, are changed. The Epm value for each curve will also be different.

Figure 7 shows the effect of changing g to change curve skewness for each function. As the g parameter decreases, the partition number coordinate of the inflection point decreases, resulting in increased skewness. For 5PGL it can be seen that the RD of the inflection point is always the same as the value for the c parameter. For 5PL, not only does asymmetry change for the curve, but also the position of the inflection point changes with respect to RD. For 5PL curves the inflection point RD ordinate increases as the level of asymmetry increases hence the increased spread of the curves compared to those generated by 5PGL.

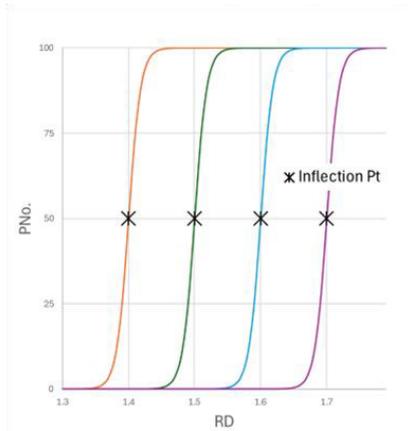
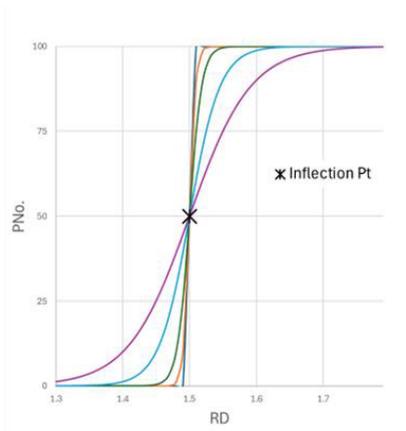


Fig. 5a. Changing rate parameter, “b”. **Fig. 5b.** Changing inflection parameter, “c”.

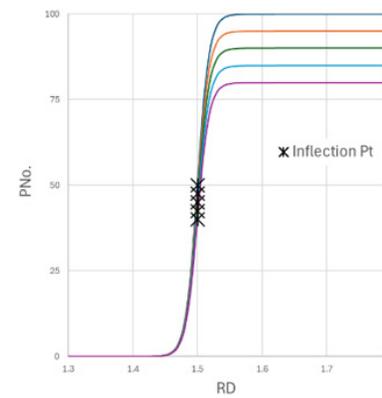
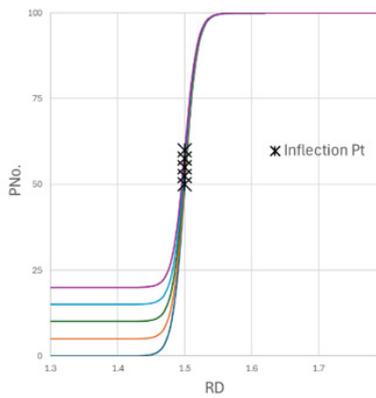


Fig. 6. Changing asymptote parameters “a” and “d” to produce “tails”.

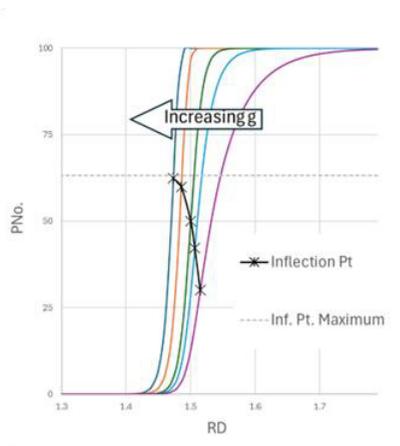
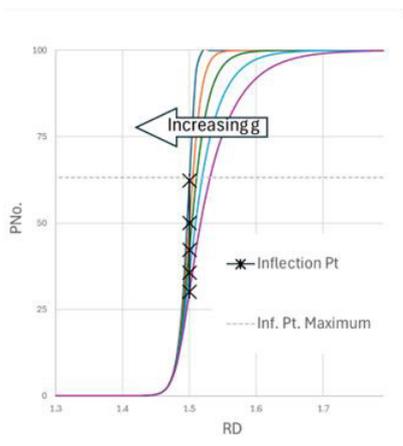


Fig. 7. Changing asymmetry parameter “g” for 5PGL and 5PL.

2.5 Managing curve form

The potential PNo range of inflection points for the curves in Figure 7 is from zero to a maximum of $(1-1/e)*100$, i.e., 63.21. That is 63.21% of the range of the curve between the lower and upper asymptotes. The horizontal line at PNo = 63.21 represents the upper limit of the inflection point. As the value of g increases, the inflection point rises but rapidly decreases as it gets closer to the maximum. Therefore, there is a limit on the level of asymmetry that the function can represent, and it may limit the quality of the fit of the curve to data, particularly if the skewness is to the product, i.e., to the left of the RD50. There is a potential solution to overcome this limiting factor. By swapping the values of a and d and changing the sign of the slope parameter, the range of the inflection point changes to 36.8 to 100. For asymmetric curves with tails, the inflection point PNo's range will need to be calculated to adjust for the reduced PNo range occupied by the curve.

As stated earlier, there are various permutations of the a and d parameters in the partition functions, and the slope parameter can be positive or negative. By changing the order of a and d , and the sign of the rate parameter b in 5PL and 5PGL, four variants emerge: two for partition to rejects and two for partition to product. Table 3, based on the work by Gottschalk and Dunn [10], shows how these variations affect partition type, curve slope, inflection point range, and major skewness. Given that there are two variants for each partition type it is necessary to record which variant has been used when curve fitting results are reported. For example, partition to rejects, skewed to product, or partition to rejects, skewed to rejects. Ideally, the equation containing the resultant parameter values would also be reported.

Table 3. Relationship between curve parameters a , d , b and curve characteristics.

Partition type	Order of a and d	Sign of b	Slope	Inflection Pt PNo Range	Major Skewness
Rejects	$a < d$	-	Up	36.8 - 100	Product
Rejects	$a > d$	+	Up	0-63.2	Rejects
Product	$a < d$	+	Down	36.8 - 100	Rejects
Product	$a > d$	-	Down	0-63.2	Product

3 Curve fitting

Curve fitting of asymmetric partition data is covered in ISO 5146 with examples given. Curves of best fit, figures 8 and 9, were calculated using the method given in the standard along with data provided in the standard from a tracer test. The method in the standard uses the beta probability distribution method to calculate a weighted residual error followed by minimizing the sum of squares of the residual errors using Microsoft's Excel Solver. The

GRG-Nonlinear method in Solver was used to vary the g , c , and b parameters within defined limits.

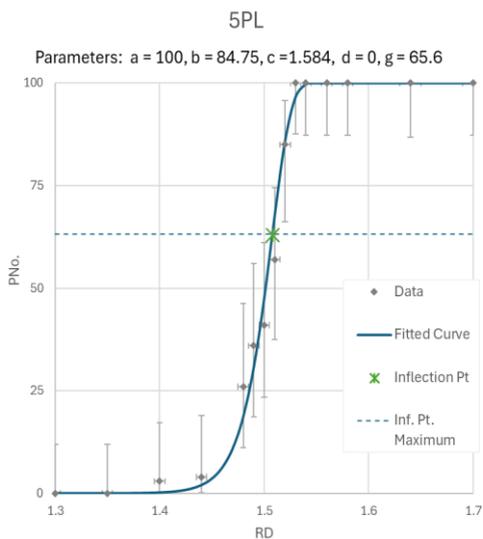


Fig. 8a. 5PL - Skewness to rejects.

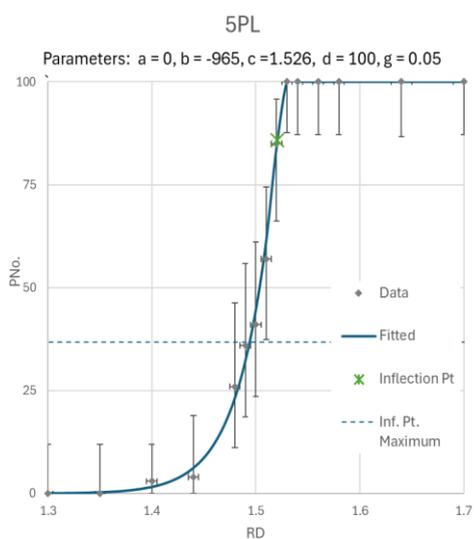


Fig. 8b. 5PL - Skewness to product.

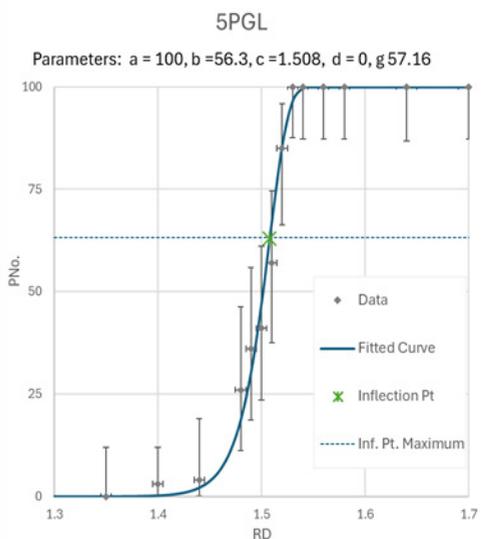


Fig. 9a. 5PGL – Skewness to rejects.

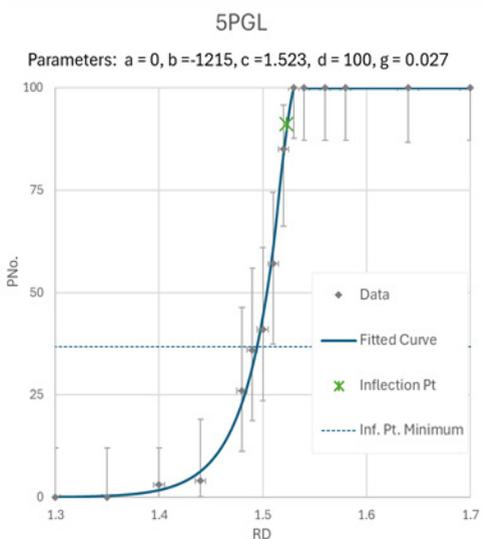


Fig. 9b. 5PGL – Skewness to product.

Figure 8a shows a curve fitted by 5PL where $a = 100$, $d = 0$, and b is positive. This matched the variant of 5PL used in ISO 5416. The error of fit of the resultant curve is 0.423, and the inflection point PNo is 63.1. The relative values of a and d parameters and the sign of the rate parameter, b , show that the function variant used was partition to rejects and skewness to rejects (Table 3.) The inflection point range upper limit is 63.2, just 0.1 greater than the PNo of the inflection point of the curve.

Curve fitting was repeated using a variant of the function suited to skewness to product for a partition to rejects curve. For the fitted curve in Figure 8b, the parameter settings were changed to $a = 0$, $d = 100$, and b was negative. These settings set the PNo. range of inflection points from 36.8 to 100. The error of fit achieved was 0.058 and the inflection point PNo was 85.8.

Visual comparison of the results suggests a better fit was obtained by the skewness to product function, especially at relative densities on the skewed side of the curve, when the range of the inflection point was 36.8-100. The value of the PNo of the point was well within the inflection point range. Whereas for the plot based on the 0-63.2 range the inflection point PNo was just 0.1 less than the maximum. This example shows that in curve fitting, if the actual inflection point is close to the inflection range limit, there is a risk of underestimating the level of asymmetry and the amount of misplaced material in densities within the region of major skewness.

Results for curve fitting by both 5PL and 5PGL are summarised in Table 5 and confirm the best fit was obtained by using skewness to product functions. It also shows that in this situation, there was no significant difference between 5PL and 5PGL. It can be concluded that to achieve the best fit to data, the type of skewness of the raw data must be assessed when selecting the form of the function to be used.

Table 4 Curve fitting results summary.

Skewness Type	5PL		5PGL	
	Product	Rejects	Product	Rejects
RD ₅₀	1.504	1.502	1.504	1.501
Epm	0.017	0.015	0.017	0.014
FA	0.236	0.509	0.214	0.497
Inf Pt.	85.8	63.1	90.8	62.89
<i>Total fit error</i>	0.067	0.424	0.065	0.308
<i>Product fit error</i>	0.053	0.139	0.053	0.154
<i>Rejects fit error</i>	0.013	0.276	0.012	0.155

4 Process performance indicators from partition curves

Table 5 lists primary and secondary process performance indicators determined by using the 5PGL function. Reporting curve fitting results when asymmetry is involved should include, as a minimum, those listed in Table 5. The process performance indicators (PPI's) are derived from the fitted curves. Table 6 compares curve fitting results for the 5PL and 5PGL for skewness type. The inflection point partition number indicates the reliability of the selected

function type with respect to skewness type. It is preferable not to have the inflection point near the limit of the range, in this case, 63.2 for skewness to reject.

Table 5. Process performance indicators (PPI's)

Process Performance Indicators	
RD50	1.504
Epm	0.017
Ep _{98/02}	0.06
Low density tail (d)	0
High density tail (a)	100
Skewness Type	Product
Asymmetry factor, Fa	0.214
Asymmetry parameter (g)	0.027
Rate parameter (b)	-1215
Inflection parameter (c)	1.523
Inflection Pt coordinates	1.523, 90.8

Table 6. PPI data

PPI Data	
Fitting function	5PGL
Curve type	To rejects
Inflection Pt PNo range	36.8 - 100
Curve fit error (total)	0.065
Low density fit error	0.053
High density fit error	0.012
Number of test data points	15

Error Area is not included among the performance indicators. Although the Error Area is a measure of separation efficiency, it is infrequently utilised. Prior to the availability of spreadsheet programs such as Microsoft Excel, the value of Error Area was not easily determined. Error Area is indicative of the overall RD range occupied by the partition curve. The larger the range, the greater the Error Area. An alternative measure is given in AS5213 [3] is $Ep_{98/02}$, equation 9, which provides the average RD range for 48% of the separation either side of the cutpoint for a partition to rejects.

$$Ep_{98/02} = \frac{(RD_{98}-RD_{02})}{2} \tag{9}$$

If the total quantity of misplaced material is required, it can be calculated using the feed washability data and respective partition numbers for the density fractions.

For asymmetric partitions, the ratio of the RD ranges 48% either side of the cutpoint partition number is a measure of asymmetry, the asymmetry factor, F_a . The factor is defined in ISO 5146, equation 10, for partition to rejects curves.

$$F_a = \frac{(RD_{98}-RD_{50})}{(RD_{50}-RD_{02})} \tag{10}$$

For partition to product curves equation 11 applies.

$$F_a = \frac{(RD_{02}-RD_{50})}{(RD_{50}-RD_{98})} \tag{11}$$

For asymmetry to product, F_a will be less than 1. The smaller the value, the higher the level of skewness. If the asymmetry is to rejects, F_a will be greater than 1 and the larger the value, the higher the level of skewness.

5 Conclusion

The five-parameter functions discussed are versatile and can be used for symmetric and asymmetric partitioning, as well as high and/or low density tails. Each function can replace the Whiten and modified Whiten functions for curve fitting, process modelling and process simulations. Using the information and guidelines presented, curve fitting of test data should be straightforward.

By modifying the rate parameter for 5PL, as given in equation 8, both 5PL and 5PGL (with $b = \ln(3)/E_{pm}$) can be utilized in existing process simulations based on Whiten functions. The inclusion of asymmetry in process simulations is not necessary unless the effects of the presence of asymmetry are being investigated. However, before asymmetry can be reliably included in simulation models, more work needs to be done to understand the relationships between asymmetry and operating variables such as cutpoint, dense medium cyclone feed pressure, particle size distribution, washability, feed solids loading, and dense medium rheology.

Employing five-parameter functions for curve fitting when asymmetry is present increases the number of process performance indicators that can be used to quantify the efficiency and effectiveness of density separation. It is particularly important also to report data associated with curve fitting results, such as fit error, and inflection point PNo range. Given that a curve could represent skewed to product or skewed to rejects scenarios, it is necessary to report the form of the curve function used, for example, “partition to rejects, skewed to product.”

While both 5PL and 5PGL can be used in simulations and curve fitting, it is likely that 5PGL will be the preferred option for curve fitting and simulations given its similarities with the Whiten functions already in use.

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