Assessment of the Influence of Boundary Conditions on the Accuracy of Impact Force Identification

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Abstract. This article focuses on the identification of impact forces acting on elastic struc-tures, a key area of structural health monitoring (SHM). The objective is to ana-lyze the effect of boundary conditions on the accuracy of force reconstruction. Structural health monitoring is an inverse problem that is both complex and gen-erally considered ill-posed. In particular, the deconvolution of measured signals is inherently unstable, requiring the application of Tikhonov regularization to obtain a reliable solution. The formulation of the forward problem is presented under the assumption of elastic supports. Using the Euler-Bernoulli model, the study leads to the development of the transfer matrix equation based on the deformation re-sponse. Simulations performed on a beam show that support stiffness plays a central role in the accuracy of force reconstruction. When the supports are too flexible, the reconstruction error is significant, especially in the presence of noise. However, beyond a certain stiffness threshold, the error stabilizes, indicating the existence of an optimal rigidity level.

Keywords: Impact Force, Identification, Regularization, SHM, Tikhonov, Re-construction.Page layout

1 Introduction

The identification of dynamic forces acting on structures is a fundamental challenge in various engineering fields. Accurate force estimation is essential for understanding structural behavior, monitoring structural health (SHM), conducting preventive maintenance, and optimizing mechanical designs. Misestimating applied forces can lead to analytical errors, reduce structural reliability, and, in extreme cases, result in catastrophic failures.

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Among structural elements, beams play a crucial role in load distribution and dissipation in civil engineering applications. Their design and analysis require meticulous attention, as even minor miscalculations can compromise the integrity of an entire structure. A key challenge in force reconstruction lies in properly accounting for boundary conditions. The dynamic response of a beam with rigid supports differs significantly from that of a beam with elastic supports, which allow for partial de-formation and energy absorption. Rigid supports impose stricter constraints on dis-placement and rotation, leading to a more predictable response, while elastic supports introduce additional complexities such as frequency shifts and damping effects.

From a mathematical perspective, impact force reconstruction is an inverse problem, often formulated as the deconvolution of two signals: the measured response and the impulse response function. However, deconvolution is inherently ill-posed, meaning small measurement errors can lead to large deviations in the reconstructed force. Direct inversion of the system is typically unreliable due to numerical instability and amplification of noise. To address this, regularization techniques are employed to stabilize the solution and obtain physically meaningful results.

Early force reconstruction methods relied on spectral analysis, utilizing the convolution theorem to express deconvolution as a division in the frequency domain [1–3]. However, this approach is highly sensitive to noise, limiting its practical application. To overcome these limitations, subsequent research [4–7] introduced regularization-based methods, particularly Tikhonov regularization [8], which stabilizes the inverse problem by incorporating a smoothing constraint. This technique is often implement-ed via singular value decomposition (SVD) or generalized singular value decomposi-tion (GSVD) of the system's Toeplitz matrix. Additionally, filtering techniques, such as truncated SVD (TSVD) or frequency-domain filtering [9], have been applied to suppress high-frequency noise components. These approaches help mitigate instability but introduce a trade-off between noise reduction and loss of fine details in the reconstructed force.

Initial studies primarily focused on localized impact forces, modeling the colliding object as a point force. However, in many real-world scenarios, impacts originate from distributed sources rather than discrete points. To address this, researchers have developed specialized methods for distributed force reconstruction in structures such as Euler-Bernoulli beams and plates [10–12]. These approaches utilize modified modal analysis and mode selection techniques to account for spatially extended forces.

Recent advancements in impact force identification have led to significant refinements in reconstruction methods. For instance, state-space models, combined with system identification (SI) and frequency-based iterative signal tuning (FIST) algorithms, have demonstrated superior accuracy compared to traditional L2-regularization techniques [13]. Additionally, finite element-based inverse approaches have been applied to simulate impact signals in composite structures, showing that, in some cases, accurate reconstruction can be achieved using only a single vibration mode [14]. Furthermore, sparse regularization techniques have been explored to ad-dress the underdetermined nature of impact force reconstruction, offering improved stability and precision in force estimation [15].

The objective of this study is to reconstruct impact forces using Tikhonov regularization to stabilize the inversion of the transfer matrix. The robustness of this method is evaluated with respect to noise levels and boundary conditions, providing insights into the influence of support stiffness on reconstruction accuracy.

2 Formulation of the Direct Problem

Let us consider a beam resting on elastic supports. It is made of a linear elastic, homogeneous, and isotropic material. The beam, with a length L, width b, and height h, has the following mechanical properties: E, the Young's modulus, and v , the Poisson's ratio. The beam is subjected to a localized impact resulting from a pressure in the form of a Dirac impulse (see Fig. 1). This pressure, denoted P(t), where t represents time, is applied at a point x_p. The dynamic response, in terms of deformations along the z-direction, is measured by a sensor located at x_s.

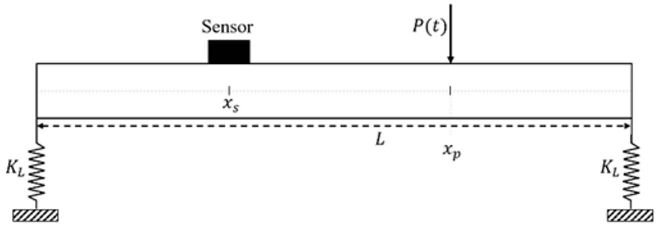

Fig. 1. An elastic beam with a uniform rectangular cross-section, elastically supported and subjected to an impact force.

Under these assumptions, the transverse displacement of the beam occurs within the vertical plane of symmetry. Considering the damping effect, the Euler-Bernoulli model governing the transient dynamic response of the beam following an impact is expressed by the following equation:

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + c\,\frac{\partial w(x,t)}{\partial t} + EI\,\frac{\partial^4 w(x,t)}{\partial x^4} = P(t) \tag{1}$$

where ρA the mass per unit length, I the second moment of inertia, and c is the damping coefficient of the beam respectively.

Eq. (1) is subjected to the following boundary conditions, expressed in terms of displacement:

$$\begin{aligned} & w_{,xx}(x=0,t) = w_{,xx}(x=L,t) = 0 \\ & EIw_{,xxx}(x=0,t) = -K_L w(0,t) \\ & EIw_{,xxx}(x=L,t) = K_L w(L,t) \end{aligned} \tag{2}$$

Since the system is linear, the modal superposition method will be applied, and the transverse displacement $w(x,t)$ can be expressed as:

$$w(x,t) = \sum_{i=1}^{\infty} \left\{ \frac{\varphi_i(x)}{M_i} \int_0^t g_i(t-\tau) P(\tau) \varphi_i\left(x_p\right) d\tau \right\} \tag{3}$$

The strain in the beam at point x_s and time t can be written as:

$$\varepsilon_x(x_s, m) = -\frac{h}{2}\frac{\partial^2 w(x,t)}{\partial x^2} \approx -\sum_{i=1}^{N} \left\{ \frac{h \Delta t \phi_i''(x_s) \varphi_i\left(x_p\right)}{2M_i} \sum_{j=0}^{m} \left[P(j\Delta t) h_i\left((m-j)\Delta t\right) \right] \right\} \tag{4}$$

with $(m = 1,2,3,\dots,N_t)$, $(j = 0,1,2,3,\dots,N_t - 1)$, and $M_i = \int_L \rho A \varphi_i^2 dx$.

where N is the number of vibration mode; φ_i is the ith mode; N_t is the number of data points and $\Delta t = \frac{T_{ob}}{N_t}$ is the time interval, and the expression of h_i is given as:

$$g_i\left[(m-j)\Delta t\right] = \begin{cases} \frac{1}{\omega_i\sqrt{1-\xi_i^2}}\exp\left[-\xi_i\omega_i(m-j)\Delta t\right]\sin\left[\omega_i\sqrt{1-\xi_i^2}\,(m-j)\Delta t\right] & \text{si } j \le m \\ 0 & \text{si } m < j \end{cases} \tag{5}$$

Eq. (6) can be rewritten in compact form as follows:

$$Y = GF \tag{6}$$

where $Y = \left[\varepsilon(\Delta t), \varepsilon(2\Delta t), ..., \varepsilon(N_t\Delta t)\right]^T$; $F = \left[P(0), P(1\Delta t), ..., P((N_t-1)\Delta t)\right]^T$ and G defined as:

$$G = \begin{bmatrix} G_{10} & 0 & \cdots & 0 \\ G_{20} & G_{21} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ G_{N_t 0} & G_{N_t 1} & \cdots & G_{N_t N_t - 1} \end{bmatrix} \tag{7}$$

where G_{mj} expression can be defined as follows:

$$G_{mj} = -\sum_{i=1}^{N}\left\{\frac{h\Delta t\varphi_i''(x_s)\varphi_i\left(x_p\right)}{2M_i} g_i\left((m-j)\Delta t\right)\right\} \tag{8}$$

3 Tikhonov Regularization

To address the ill-posed nature of an impact force reconstruction problem, it is necessary to use a regularization method. This approach involves introducing additional physical or mathematical constraints to render the problem well-posed. In this work, Tikhonov regularization has been adopted [8]. This method is widely used, particularly in the context of deconvolution problems. Jacquelin et al. [6] notably presented its application in such cases. Tikhonov regularization introduces a norm designed to enforce smoothing in the least-squares framework. In the context of a general deconvolution problem, the Tikhonov regularization least squares formulation is expressed as follows:

$$\min_F\left[\left\|[G]\{F\} - [Y]\right\|_2 + \beta\left\|[I]\{F\}\right\|_2\right] \tag{9}$$

Here, $[G]$ is the transfer matrix obtained by discretizing the convolution integral through sampling as described in Eq. (9), $\{F\}$ is load vector, $[Y]$ is the measured response, $\|\bullet\|_2$ denotes the Euclidean norm, and β is the regularization parameter.

4 Simulation and Results

The structure of Fig. 1 is assumed to have the following dimensions: L=0.9m, b=0.02m and h= 0.02m; elastic supports stiffness K_L=1000N/m. The material properties of the aluminum beam were given as: E = 70 GPa, υ=0.33 , masse density ρ = 2800 kg/m³, and a modal damping ratio $\xi = 0.02$ applicable to all modes. The impact pressure signal, defined as a half-sine wave, is represented over the interval [0,0.5] s , with a period T=0.2 s and Δt =0.001s . The impact signal profile considered in this work admits the form of a half-sine function is shown in Fig. 2, applied at point x_p=0.7m.

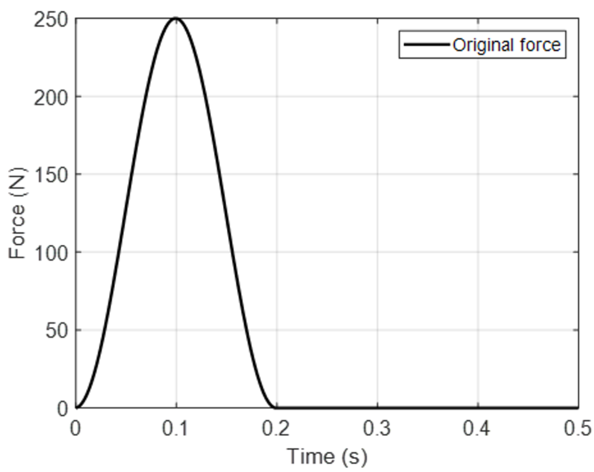

Fig.2. Impact signal profile defined with $T = 0.2s$

4.1 Effect of Support Stiffness on the Dynamic Response

To assess the influence of support elasticity, a comparative analysis was conducted considering three support cases: 1) rigid supports, 2) elastic supports with moderate stiffness, and 3) elastic supports with very low stiffness, which is similar to the free edges of a beam. This analysis aimed to highlight the impact of elastic supports on the reconstruction of the applied load. Fig. 3 illustrates the dynamic responses of the three considered cases, with a sensor placed at x_s=0.3m.

As shown in Fig. 3, support rigidity significantly affects the measured deformation. For the classical case, it is observed that, in this linear system, the response initially mirrors the applied signal. However, as time progresses, the response deviates from the applied signal due to the elasticity of the support. This highlights the influence of support elasticity on the system's dynamic behavior, showing how increasing flexibility in the supports distorts the signal's shape in the measured response.

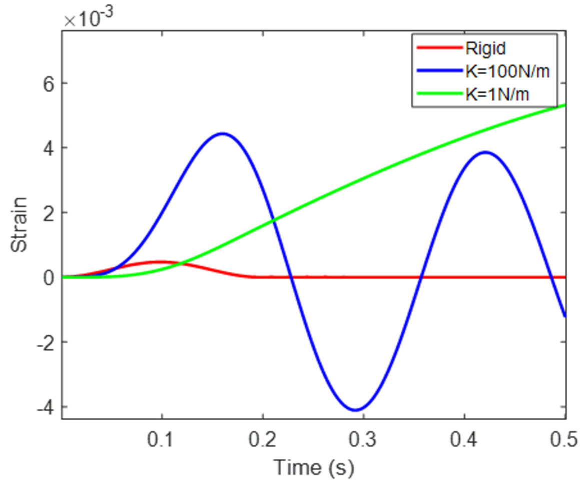

Fig. 3. The Strain as a function of time for varied value of K_L

4.2 Effect of Support Stiffness and Noise Level on the Reconstruction Force

The reconstruction of the impact force requires the use of a regularization method. In this work, we have used the Tikhonov method to improve the quality of the reconstruction. In this context, the robustness of this technique will be evaluated based on the noise level and the support stiffness simultaneously. White noise was added to the direct elastic response, expressed in terms of strain, in order to store it for later use in impact force reconstruction. Noise levels of 1%, 3%, and 5% are examined separately.

$$Y_{Noised} = Y + E_p \sigma(Y) \quad (10)$$

where Y_{Noised} signifies the strain and E_p denotes the noise level. The noise is characterized as a standard normal distribution vector, exhibiting a mean of zero and a standard deviation of one. Y indicates the computed strain, whereas $\sigma(Y)$ denotes the corresponding standard deviations.

Fig. 4 presents the reconstruction of the impact force for three different noise levels, highlighting the influence of support stiffness on the identification process. At a low noise level (see Fig. 4a), the reconstructed force closely matches the original force, particularly in the case of the rigid support, where the solution remains highly accurate. As the noise level increases (Fig. 4b), deviations from the exact solution become more pronounced. Notably, at a noise level of 5% (see Fig. 4c), the reconstruction for the rigid support remains nearly perfect, demonstrating the robustness of the Tikhonov method in this scenario.

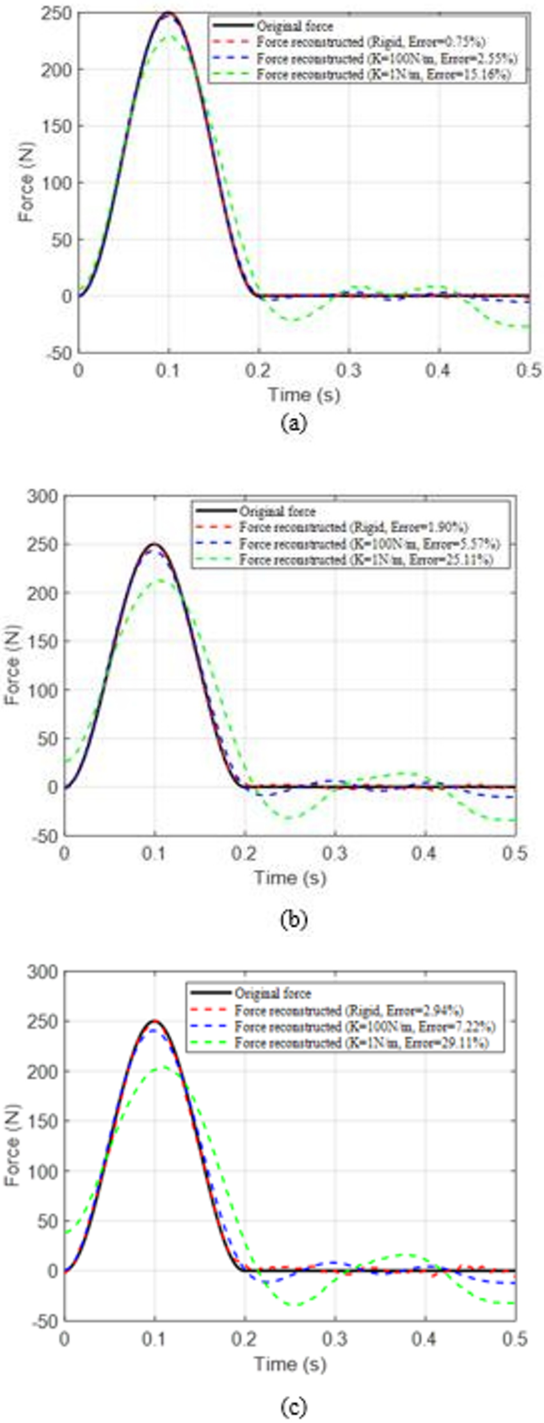

Fig. 4. Effect of support stiffness on the accuracy of impact force reconstruction under a given noise level; a) level noise 1%, b) level noise 3%, c) level noise 5%.

However, the reconstruction quality deteriorates as the support stiffness decreases, leading to significant errors for the elastic support. In particular, higher noise levels amplify discrepancies in the identified force, with the elastic support exhibiting pronounced errors, whereas the rigid support maintains a relatively stable and acceptable solution.

4.3 Relative Error

The robustness of the Tikhonov regularization method and the accuracy of the im-pact force reconstruction will be assessed by computing the relative errors associated with each noise level. The relative error quantifies the deviation between the recon-structed and actual impact forces, providing a measure of reconstruction quality. This error is calculated using the following formula:

$$Error = \frac{\| F_{reg} - F_{real} \|}{\| F_{real} \|} \times 100 \tag{11}$$

Fig. 5 demonstrates the fluctuation in relative error in relation to support stiffness K_L for various noise levels (1%, 3% and 5%). It is noticed that the error is initially considerable for low values of K_L, especially at high noise levels, before dropping fast and stabilizing for increasing stiffnesses. This pattern implies that the use of soft supports increases the influence of noise, leading to a reduction in the accuracy of the results. However, beyond a value of 1000 N/m, the error becomes steadier and less reactive to disturbances. These results stress the necessity of appropriate stiffness to boost the robustness of identification, while highlighting the effect of noise on measurement accuracy. It is important to note that, in the case of a rigid support and a noise level of 5%, the Tikhonov regularization method demonstrates high efficiency and robustness in the reconstruction process, with a relative error of only 2.94%. However, for the same noise level, the relative error increases to 7.22% for a support with intermediate stiffness (100 N/m), and becomes significant for a low-stiffness support (1 N/m), reaching 29.11%.

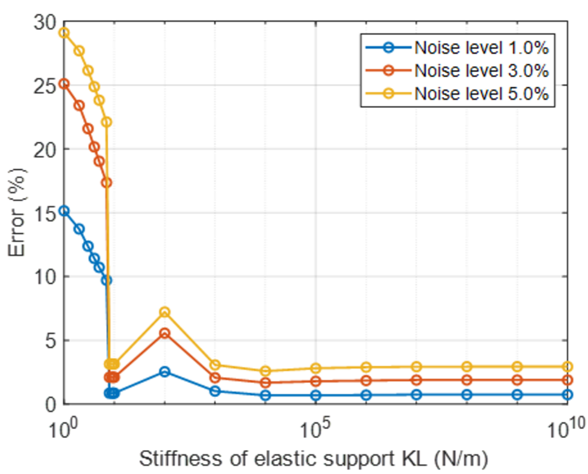

Fig. 5. Evolution of the error as a function of stiffness with a noise level varied from 1% to 5%.

5 Conclusion

This work focuses on the identification of impact forces as part of structural health monitoring, through the reconstruction of the external force applied to a beam. The study examines the influence of key parameters on the accuracy of this reconstruction, highlighting the robustness of the Tikhonov regularization method under varying noise levels and support stiffness. The results emphasize the crucial role of boundary conditions in the identification process, especially in the presence of noise. They also show that support stiffness significantly affects reconstruction accuracy: for very low stiffness values (1 N/m), the errors become substantial, reaching 29.11%, which com-promises the reliability of the results. However, beyond a certain stiffness threshold, around 1000 N/m, the error stabilizes and remains below 5%, ensuring more robust and consistent reconstruction.

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